Parallel Scientific Computing

Course AMS301 — Fall 2023 — Lecture 8

Introduction to domain decomposition methods (DDM)

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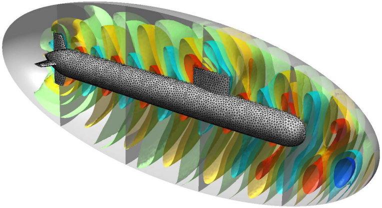
DD methods — Context and motivation [1/3]

Context

- Parallel solution of problems defined with partial differential equations (PDE) and discretized with finite differences, finite elements, finite volumes, . . .
- Sparse linear system to be solved *(elliptic problems; hyperbolic and/or parabolic problems with implicit time stepping)*

Classical approaches:	Direct method: expensive and difficult to parallelize
	Iterative method: slow convergence

Aeroacoustics
[Boubendir, Antoine, Geuzaine, 2012]



Wave propagation in geological structure
[Vion & Geuzaine, 2017]

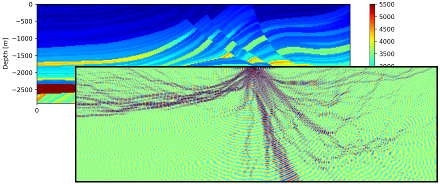


Illustration with a 1D discrete problem

Continuous problem:

$$\left| \begin{array}{l} -u'' = f, \quad \text{for } x \in]0,1[\\ u(0) = g_L \quad \text{and} \quad u(1) = g_R \end{array} \right.$$

Discrete problem:

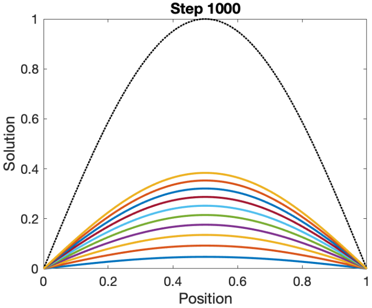
$$\left| \begin{array}{l} \dfrac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} = f_i, \quad i = 1, \dots, N \\ u_0 = g_L \quad \text{and} \quad u_{N+1} = g_R \end{array} \right.$$

Algebraic system:

$$\underbrace{\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & \ddots & & & & & \\ & \ddots & \ddots & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & \ddots & \\ & & & & & \ddots & \ddots & -1 \\ & & & & & & -1 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{i-1} \\ u_i \\ u_{i+1} \\ \vdots \\ u_n \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} h^2 f_1 + g_L \\ h^2 f_2 \\ \vdots \\ h^2 f_{i-1} \\ h^2 f_i \\ h^2 f_{i+1} \\ \vdots \\ h^2 f_n + g_R \end{bmatrix}}_{\mathbf{b}}$$

Illustration with a 1D discrete problem

Jacobi method



Gauss-Seidel method

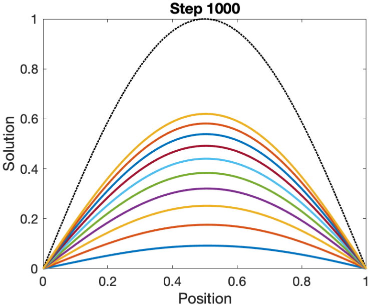


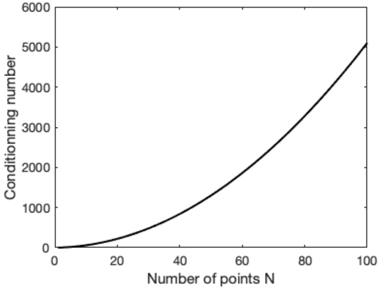
Illustration with a 1D discrete problem

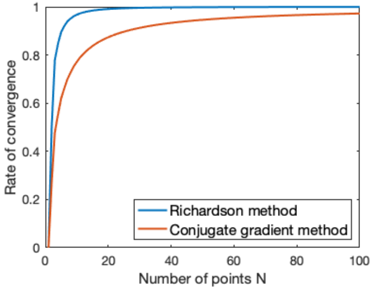
Convergence rate of iterative methods by points: *(for information)*

Richardson method: $$\frac{\|\mathbf{e}^{(\ell+1)}\|}{\|\mathbf{e}^{(\ell)}\|} \leq \frac{\kappa(\mathbf{A})-1}{\kappa(\mathbf{A})+1} \simeq 1-\mathcal{O}(h^2)$$

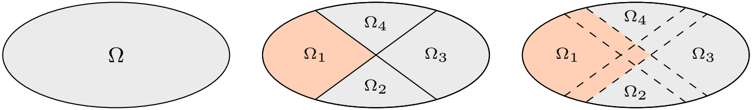
Conjugate gradient method: $$\frac{\|\mathbf{e}^{(\ell+1)}\|}{\|\mathbf{e}^{(\ell)}\|} \leq \frac{\sqrt{\kappa(\mathbf{A})}-1}{\sqrt{\kappa(\mathbf{A})}+1} \simeq 1-\mathcal{O}(h)$$

with
- error $\mathbf{e}^{(\ell)} := \mathbf{u}^{(\ell)} - \mathbf{u}^{\text{ref}}$
- condition number $\kappa(\mathbf{A}) := \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2 \simeq 4/(\pi^2 h^2)$.





Original domain | Partition without overlap | Partition with overlap



Partition without overlap: $\overline{\Omega} = \overline{\cup_I \Omega_I}$, $\Omega_I \cap \Omega_J = \emptyset$ if $I \neq J$

Partition with overlap: $\overline{\Omega} = \overline{\cup_I \Omega_I}$

Basic concepts on domain decomposition methods:

- *Partition* of domain Ω into *subdomains* $\{\Omega_I\}_I$ *(with or without overlap)*
- Parallel solution of *subproblems* defined on subdomains $\{\Omega_I\}_I$
- Techniques for *coupling the solution at the interfaces* between the subdomains
- *Preconditioning* techniques to speed up the global iterative scheme

Goal for Today: *Speed up the convergence of iterative solution procedures*
Introduction to fundamental domain decomposition methods

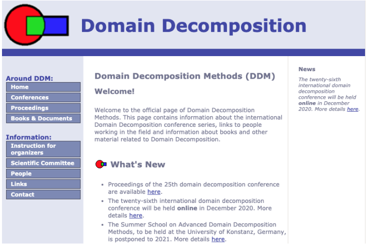
DD methods — A very active topic of research . . .

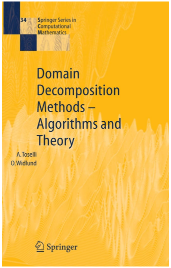
Topic at the crossroads of mathematics, engineering and computer science.

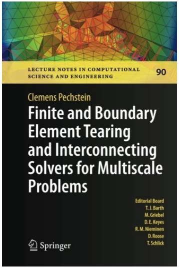
References

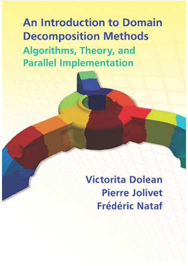
- *Méthodes Num. : Algos., analyse et appli.*, de Quarteroni, Sacco, Saleri (2007), Springer
- *An into. to DD Methods*, de Dolean, Jolivet, Nataf (2015), SIAM `[lien]`
- Slides de V. Martin *(LAMFA, U. de Picardie Jules Verne)* `[lien]`

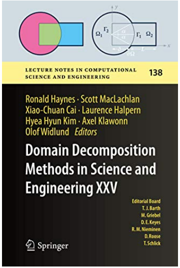
`http://www.ddm.org`











Introduction to domain decomposition methods (DDM)

- *Block stationary methods* *(discrete ⊙)*
- *Schwarz methods* *(continuous ⊙)*
- *Parallel preconditioning* *(discrete ⊙)*

Illustration with a 1D discrete problem

Continuous problem:

$$\left| \begin{array}{l} -u'' = f, \quad \text{for } x \in]0,1[\\ u(0) = g_L \quad \text{and} \quad u(1) = g_R \end{array} \right.$$

Discrete problem:

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Algebraic system:

$$\underbrace{\begin{bmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & \ddots & & & & & & \\ & \ddots & \ddots & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & \ddots & & \\ & & & & & \ddots & \ddots & -1 \\ & & & & & & -1 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{i-1} \\ u_i \\ u_{i+1} \\ \vdots \\ u_n \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} h^2 f_1 + g_L \\ h^2 f_2 \\ \vdots \\ h^2 f_{i-1} \\ h^2 f_i \\ h^2 f_{i+1} \\ \vdots \\ h^2 f_n + g_R \end{bmatrix}}_{\mathbf{b}}$$

Stationary method: $\boxed{\mathbf{M}\mathbf{u}^{(\ell+1)} = \mathbf{N}\mathbf{u}^{(\ell)} + \mathbf{b}}$ with $\mathbf{A} = \mathbf{M} - \mathbf{N}$ and $\mathbf{M}$ non-singular

Block partition of the matrix and the vectors:

$$\mathbf{A} = \left[\begin{array}{cccc|cccc} 2 & -1 & & & & & & \\ -1 & 2 & \ddots & & & & & \\ & \ddots & \ddots & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ \hline & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & \ddots & \\ & & & & & \ddots & \ddots & -1 \\ & & & & & & -1 & 2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$$

$$\mathbf{u} = \left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{u}_2 \end{array}\right] \qquad \mathbf{b} = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \end{array}\right]$$

Choices for $\mathbf{M}$:

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{A}_{11} & 0 \\ \hline 0 & \mathbf{A}_{22} \end{array}\right]$$ *(Block Jacobi)*

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{A}_{11} & 0 \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$$ *(Block Gauss-Seidel)*

Considered problem:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

Block Jacobi method:

$$\begin{bmatrix} \mathbf{A}_{11} & 0 \\ 0 & \mathbf{A}_{22} \end{bmatrix} \mathbf{u}^{(\ell+1)} = \begin{bmatrix} 0 & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & 0 \end{bmatrix} \mathbf{u}^{(\ell)} + \mathbf{b}$$

$$\Leftrightarrow \quad \begin{cases} \mathbf{A}_{11}\mathbf{u}_1^{(\ell+1)} = -\mathbf{A}_{12}\mathbf{u}_2^{(\ell)} + \mathbf{b}_1 \\ \mathbf{A}_{22}\mathbf{u}_2^{(\ell+1)} = -\mathbf{A}_{21}\mathbf{u}_1^{(\ell)} + \mathbf{b}_2 \end{cases}$$

$$\Leftrightarrow \quad \boxed{\begin{cases} \mathbf{u}_1^{(\ell+1)} = \mathbf{u}_1^{(\ell)} + \mathbf{A}_{11}^{-1}\left(\mathbf{b}_1 - \mathbf{A}_{11}\mathbf{u}_1^{(\ell)} - \mathbf{A}_{12}\mathbf{u}_2^{(\ell)}\right) \\ \mathbf{u}_2^{(\ell+1)} = \mathbf{u}_2^{(\ell)} + \mathbf{A}_{22}^{-1}\left(\mathbf{b}_2 - \mathbf{A}_{21}\mathbf{u}_1^{(\ell)} - \mathbf{A}_{22}\mathbf{u}_2^{(\ell)}\right) \end{cases}}$$

Remark: Solving $\mathbf{v} = \mathbf{A}_{II}^{-1}\widetilde{\mathbf{b}}$ $\Leftrightarrow$ Solving $\mathbf{A}_{II}\mathbf{v} = \widetilde{\mathbf{b}}$

$\Leftrightarrow$ Solving a PDE problem on Ω_I

Considered problem:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

Block Gauss-Seidel method:

$$\begin{bmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{u}^{(\ell+1)} = \begin{bmatrix} 0 & -\mathbf{A}_{12} \\ 0 & 0 \end{bmatrix}$$

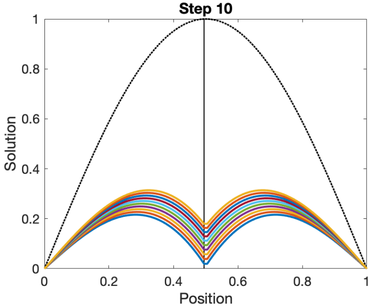
$$\Leftrightarrow \quad \begin{cases} \mathbf{A}_{11}\mathbf{u}_1^{(\ell+1)} = -\mathbf{A}_{12}\mathbf{u}_2^{(\ell)} + \mathbf{b}_1 \\ \mathbf{A}_{22}\mathbf{u}_2^{(\ell+1)} = -\mathbf{A}_{21}\mathbf{u}_1^{(\ell+1)} + \mathbf{b}_2 \end{cases}$$

$$\Leftrightarrow \quad \boxed{\begin{cases} \mathbf{u}_1^{(\ell+1)} = \mathbf{u}_1^{(\ell)} + \mathbf{A}_{11}^{-1}\left(\mathbf{b}_1 - \mathbf{A}_{11}\mathbf{u}_1^{(\ell)} - \mathbf{A}_{12}\mathbf{u}_2^{(\ell)}\right) \\ \mathbf{u}_2^{(\ell+1)} = \mathbf{u}_2^{(\ell)} + \mathbf{A}_{22}^{-1}\left(\mathbf{b}_2 - \mathbf{A}_{21}\mathbf{u}_1^{(\ell+1)} - \mathbf{A}_{22}\mathbf{u}_2^{(\ell)}\right) \end{cases}}$$

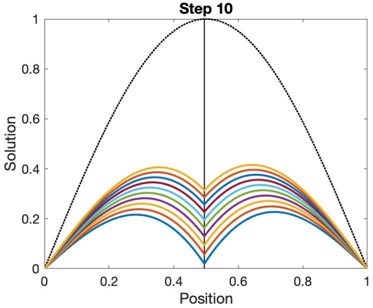
Remark: Solving $\mathbf{v} = \mathbf{A}_{II}^{-1}\widetilde{\mathbf{b}}$ $\Leftrightarrow$ Solving $\mathbf{A}_{II}\mathbf{v} = \widetilde{\mathbf{b}}$
$\Leftrightarrow$ Solving a PDE problem on Ω_I

Illustration with a 1D discrete problem

Block Jacobi method



Block Gauss-Seidel method



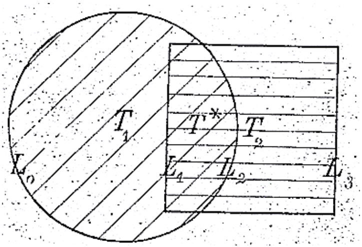
Introduction to domain decomposition methods (DDM)

- *Block stationary methods* *(discrete $\odot$)*
- *Schwarz methods* *(continuous $\odot$)*
- *Parallel preconditioning* *(discrete $\odot$)*

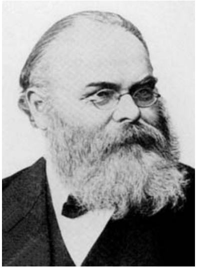
H. A. SCHWARZ, *Ueber einen Grenzübergang durch alternirendes Verfahren*, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, 15 (1870), pp. 272–286

We consider the Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

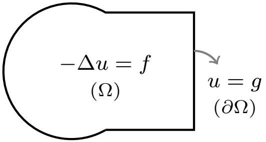


(picture from the original article)

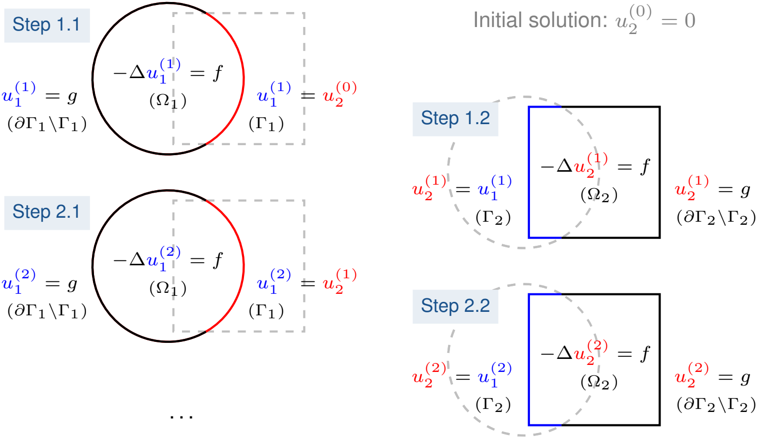


Hermann Amandus Schwarz (1843-1921)

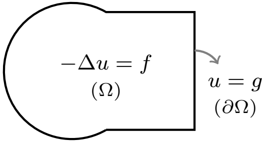
Considered problem:



Solution procedure with the Schwarz alternating method *[Schwarz, 1870]*:



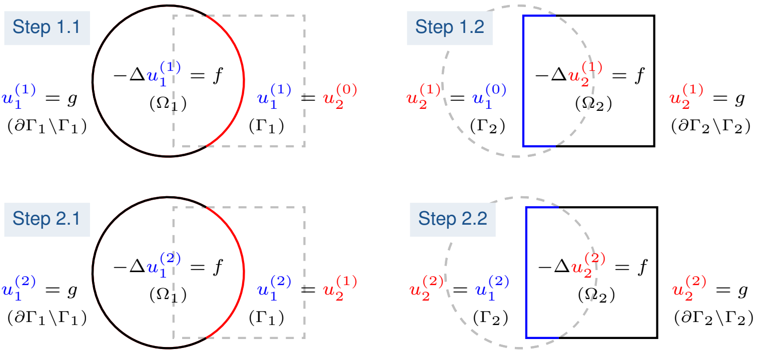
Considered problem:



Solution procedure with the Schwarz parallel method *[Lions, 1988]*:

Initial solution: $u_1^{(0)} = 0$

Initial solution: $u_2^{(0)} = 0$



... ...

Schwarz methods — Standard schemes [1/2]

Schwarz alternating method *[Schwarz, 1870]*

$$\begin{cases} -\Delta u_1^{(\ell+1)} = f & \text{in } \Omega_1 \\ u_1^{(\ell+1)} = u_2^{(\ell)} & \text{in } \Gamma_1 \\ u_1^{(\ell+1)} = g & \text{in } \partial\Omega_1 \backslash \Gamma_1 \end{cases} \qquad \begin{cases} -\Delta u_2^{(\ell+1)} = f & \text{in } \Omega_2 \\ u_2^{(\ell+1)} = u_1^{(\ell+1)} & \text{in } \Gamma_2 \\ u_2^{(\ell+1)} = g & \text{in } \partial\Omega_2 \backslash \Gamma_2 \end{cases}$$

Schwarz parallel method *[Lions, 1988]*

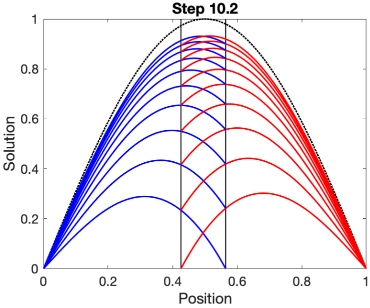
$$\begin{cases} -\Delta u_1^{(\ell+1)} = f & \text{in } \Omega_1 \\ u_1^{(\ell+1)} = u_2^{(\ell)} & \text{in } \Gamma_1 \\ u_1^{(\ell+1)} = g & \text{in } \partial\Omega_1 \backslash \Gamma_1 \end{cases} \qquad \begin{cases} -\Delta u_2^{(\ell+1)} = f & \text{in } \Omega_2 \\ u_2^{(\ell+1)} = u_1^{(\ell)} & \text{in } \Gamma_2 \\ u_2^{(\ell+1)} = g & \text{in } \partial\Omega_2 \backslash \Gamma_2 \end{cases}$$

The Schwarz parallel method can be parallelized,
but the convergence is slower than the Schwarz alternating method.

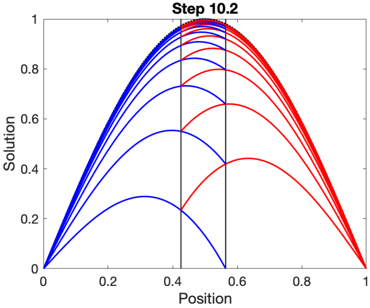
These methods converge only if there is an overlap
between the subdomains Ω_1 and Ω_2, i.e. $\Omega_1 \cap \Omega_2 \neq \emptyset$.

Illustration with a 1D discrete problem

Schwarz parallel method



Schwarz alternating method



Schwarz methods — Alternative approaches ...

Dirichlet-Neumann method *[Bjørstad, Widlund 1986]*

$$\begin{cases} -\Delta u_1^{(\ell+1)} = f & \text{in } \Omega_1 \\ u_1^{(\ell+1)} = u_2^{(\ell)} & \text{in } \Gamma_1 \\ u_1^{(\ell+1)} = g & \text{in } \partial\Omega_1 \backslash \Gamma_1 \end{cases} \qquad \begin{cases} -\Delta u_2^{(\ell+1)} = f & \text{in } \Omega_2 \\ \partial_{n_2} u_2^{(\ell+1)} = \partial_{n_2} u_1^{(\ell+1)} & \text{in } \Gamma_2 \\ u_2^{(\ell+1)} = g & \text{in } \partial\Omega_2 \backslash \Gamma_2 \end{cases}$$

Optimized Schwarz method *(with impedance operator $\mathcal{R}_i = \partial_{n_i} + \alpha_i$)*

$$\begin{cases} -\Delta u_1^{(\ell+1)} = f & \text{in } \Omega_1 \\ \mathcal{R}_1 u_1^{(\ell+1)} = \mathcal{R}_1 u_2^{(\ell)} & \text{in } \Gamma_1 \\ u_1^{(\ell+1)} = g & \text{in } \partial\Omega_1 \backslash \Gamma_1 \end{cases} \qquad \begin{cases} -\Delta u_2^{(\ell+1)} = f & \text{in } \Omega_2 \\ \mathcal{R}_2 u_2^{(\ell+1)} = \mathcal{R}_2 u_1^{(\ell+1)} & \text{in } \Gamma_2 \\ u_2^{(\ell+1)} = g & \text{in } \partial\Omega_2 \backslash \Gamma_2 \end{cases}$$

Possible alternatives and extensions:

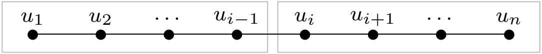
- Alternating or parallel versions *(i.e. sequential/parallel solution of subproblems)*
- Domain partition with or without overlap *(i.e. $\Omega_1 \cap \Omega_2 \neq \emptyset$ or $\Omega_1 \cap \Omega_2 = \emptyset$)*
- Different coupling techniques or coupling conditions at the interfaces
- Extensions for more than 2 subdomains

Introduction to domain decomposition methods (DDM)

- *Block stationary methods* *(discrete ⊙)*
- *Schwarz methods* *(continuous ⊙)*
- *Parallel preconditioning* *(discrete ⊙)*

Preconditioning — Rewriting the block stationary methods [1/3]

Restriction operator *(definition for a partition “without overlap”)*



$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

Restriction operator: $\boxed{\mathbf{R}_I \ : \ \mathbf{u} \ \longrightarrow \ \mathbf{R}_I\,\mathbf{u} = \mathbf{u}_I}$

Example:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_1\,\mathbf{u} = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top = \mathbf{u}_1$$

$$\mathbf{R}_2\,\mathbf{u} = \begin{bmatrix} 3 & 4 \end{bmatrix}^\top = \mathbf{u}_2$$

$$\mathbf{R}_1^\top\,\mathbf{u}_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix}^\top$$

$$\mathbf{R}_2^\top\,\mathbf{u}_2 = \begin{bmatrix} 0 & 0 & 3 & 4 \end{bmatrix}^\top$$

Key properties:

$$\mathbf{A}_{IJ} = \mathbf{R}_I \mathbf{A} \mathbf{R}_J^\top \qquad \mathbf{A} = \sum_I \sum_J \mathbf{R}_I^\top \mathbf{A}_{IJ} \mathbf{R}_J \qquad \mathbf{R}_2^\top \mathbf{A}_{21} \mathbf{R}_1 = \begin{bmatrix} 0 & 0 \\ \mathbf{A}_{21} & 0 \end{bmatrix}$$

Block Jacobi method:

$$\begin{cases} \mathbf{u}_1^{(\ell+1)} = \mathbf{u}_1^{(\ell)} + \mathbf{A}_{11}^{-1} \left(\mathbf{b}_1 - \mathbf{A}_{11}\mathbf{u}_1^{(\ell)} - \mathbf{A}_{12}\mathbf{u}_2^{(\ell)} \right) \\ \mathbf{u}_2^{(\ell+1)} = \mathbf{u}_2^{(\ell)} + \mathbf{A}_{22}^{-1} \left(\mathbf{b}_2 - \mathbf{A}_{21}\mathbf{u}_1^{(\ell)} - \mathbf{A}_{22}\mathbf{u}_2^{(\ell)} \right) \end{cases}$$

$$\Leftrightarrow \quad \mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} + \begin{bmatrix} \mathbf{A}_{11}^{-1} & 0 \\ 0 & \mathbf{A}_{22}^{-1} \end{bmatrix} \left(\mathbf{b} - \mathbf{A}\mathbf{u}^{(\ell)} \right)$$

$$\Leftrightarrow \quad \boxed{\mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} + \left[\sum_{I=1}^{2} \mathbf{R}_I^\top \mathbf{A}_{II}^{-1} \mathbf{R}_I \right] \left(\mathbf{b} - \mathbf{A}\mathbf{u}^{(\ell)} \right)}$$

Block Gauss-Seidel method:

$$\begin{cases} \mathbf{u}_1^{(\ell+1)} = \mathbf{u}_1^{(\ell)} + \mathbf{A}_{11}^{-1} \left(\mathbf{b}_1 - \mathbf{A}_{11}\mathbf{u}_1^{(\ell)} - \mathbf{A}_{12}\mathbf{u}_2^{(\ell)} \right) \\ \mathbf{u}_2^{(\ell+1)} = \mathbf{u}_2^{(\ell)} + \mathbf{A}_{22}^{-1} \left(\mathbf{b}_2 - \mathbf{A}_{21}\mathbf{u}_1^{(\ell+1)} - \mathbf{A}_{22}\mathbf{u}_2^{(\ell)} \right) \end{cases}$$

$$\Leftrightarrow \quad \cdots$$

$$\Leftrightarrow \quad \boxed{\mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} + \left[\mathbf{I} - \prod_{I=1}^{2} \left(\mathbf{I} - \mathbf{R}_I^\top \mathbf{A}_{II}^{-1} \mathbf{R}_I \mathbf{A} \right) \mathbf{A}^{-1} \right] \left(\mathbf{b} - \mathbf{A}\mathbf{u}^{(\ell)} \right)}$$

Remark: These schemes can be written as $\mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} + \mathbf{P}^{-1}\mathbf{r}^{(\ell)}$

The Jacobi and Gauss-Seidel schemes can be written as:

$$\boxed{\mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} + \mathbf{P}^{-1}\mathbf{r}^{(\ell)}}$$

with

$$\mathbf{P}^{-1} = \sum_I \mathbf{R}_I^\top \mathbf{A}_{II}^{-1} \mathbf{R}_I \qquad \textit{(Block Jacobi)}$$

$$\mathbf{P}^{-1} = \mathbf{I} - \prod_I \left(\mathbf{I} - \mathbf{R}_I^\top \mathbf{A}_{II}^{-1} \mathbf{R}_I \mathbf{A}\right) \mathbf{A}^{-1} \qquad \textit{(Block Gauss-Seidel)}$$

The scheme can be rewritten as:

$$\begin{aligned}
\mathbf{u}^{(\ell+1)} &= \mathbf{u}^{(\ell)} + \mathbf{P}^{-1}\left(\mathbf{b} - \mathbf{A}\mathbf{u}^{(\ell)}\right) \\
&= \mathbf{u}^{(\ell)} + \left(\mathbf{P}^{-1}\mathbf{b} - \mathbf{P}^{-1}\mathbf{A}\mathbf{u}^{(\ell)}\right) \\
&= \mathbf{u}^{(\ell)} + \left(\widetilde{\mathbf{b}} - \widetilde{\mathbf{A}}\mathbf{u}^{(\ell)}\right) \quad \text{avec} \quad \widetilde{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A} \ \text{ and } \ \widetilde{\mathbf{b}} = \mathbf{P}^{-1}\mathbf{b}
\end{aligned}$$

$$\Leftrightarrow \quad \boxed{\mathbf{u}^{(\ell+1)} = \mathbf{u}^{(\ell)} + \widetilde{\mathbf{r}}^{(\ell)}} \quad \text{avec } \widetilde{\mathbf{r}} = \widetilde{\mathbf{b}} - \widetilde{\mathbf{A}}\mathbf{u}^{(\ell)}$$

Solving $\mathbf{A}\mathbf{u} = \mathbf{b}$ with a Jacobi or Gauss-Seidel scheme is equivalent to solving $\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{b}}$ with the fixed-point scheme $\mathbf{u}^{(\ell+1)} = (\mathbf{I} - \widetilde{\mathbf{A}})\mathbf{u}^{(\ell)} - \widetilde{\mathbf{b}}$.

To solve $\mathbf{A}\mathbf{u} = \mathbf{b}$, we consider the equivalent problem $\mathbf{P}^{-1}\mathbf{A}\mathbf{u} = \mathbf{P}^{-1}\mathbf{b}$.
The conditioning of $\mathbf{P}^{-1}\mathbf{A}$ should be better than the one of $\mathbf{A}$, e.g. $\kappa(\mathbf{P}^{-1}\mathbf{A}) \ll \kappa(\mathbf{A})$.
Applying $\mathbf{P}^{-1}$ to any vector should be efficient.

Preconditioned conjugate gradient method *(for information)*

$\mathbf{x}^{(0)} \in \mathbb{R}^n$
$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(0)}$
$\mathbf{z}^{(0)} = \mathbf{P}^{-1}\mathbf{r}^{(0)}$
$\mathbf{p}^{(0)} = \mathbf{r}^{(0)}$
for $\ell = 0, 1, \ldots$ **do**
- $\alpha^{(\ell)} = \left(\mathbf{r}^{(\ell)}, \mathbf{z}^{(\ell)}\right) / \left(\mathbf{A}\mathbf{p}^{(\ell)}, \mathbf{p}^{(\ell)}\right)$ — Comput. of step
- $\mathbf{x}^{(\ell+1)} = \mathbf{x}^{(\ell)} + \alpha^{(\ell)}\mathbf{p}^{(\ell)} \quad = \mathbf{x}^{(0)} + \sum_{l=0}^{\ell} \alpha^{(l)}\mathbf{p}^{(l)}$ — Update
- $\mathbf{r}^{(\ell+1)} = \mathbf{r}^{(\ell)} - \alpha^{(\ell)}\mathbf{A}\mathbf{p}^{(\ell)} \quad = \mathbf{b} + \mathbf{A}\mathbf{x}^{(\ell+1)}$ — Comput. of residual
- $\mathbf{z}^{(\ell+1)} = \mathbf{P}^{-1}\mathbf{r}^{(\ell+1)}$
- $\beta^{(\ell)} = \left(\mathbf{r}^{(\ell+1)}, \mathbf{z}^{(\ell+1)}\right) / \left(\mathbf{r}^{(\ell)}, \mathbf{z}^{(\ell)}\right)$
- $\mathbf{p}^{(\ell+1)} = \mathbf{z}^{(\ell+1)} + \beta^{(\ell)}\mathbf{p}^{(\ell)}$ — Comput. of direction
- **if** $\left\|\mathbf{r}^{(\ell+1)}\right\| \leq \varepsilon \left\|\mathbf{r}^{(0)}\right\|$ **then break**

end

To solve $\mathbf{A}\mathbf{u} = \mathbf{b}$, we consider the equivalent problem $\mathbf{P}^{-1}\mathbf{A}\mathbf{u} = \mathbf{P}^{-1}\mathbf{b}$.
The conditioning of $\mathbf{P}^{-1}\mathbf{A}$ should be better than the one of $\mathbf{A}$, e.g. $\kappa(\mathbf{P}^{-1}\mathbf{A}) \ll \kappa(\mathbf{A})$.
Applying $\mathbf{P}^{-1}$ to any vector should be efficient.

Let us note that the explicit knowledge of $\mathbf{P}$ or $\mathbf{P}^{-1}$ is not required!

One can use . . .

$$\mathbf{P}^{-1} = \sum_I \mathbf{R}_I^\top \mathbf{A}_{II}^{-1} \mathbf{R}_I \qquad \text{(Additive Schwarz method)}$$

$$\mathbf{P}^{-1} = \mathbf{I} - \prod_I \left(\mathbf{I} - \mathbf{R}_I^\top \mathbf{A}_{II}^{-1} \mathbf{R}_I \mathbf{A}\right) \mathbf{A}^{-1} \qquad \text{(Multiplicative Schwarz method)}$$

and, then, using a standard linear scheme on the preconditioned system.

Summary

- **Block stationnary iterative methods** *(discrete point of view)*
  - Block Jacobi
  - Block Gauss-Seidel
- **Schwarz iterative methods** *(continuous point of view)*
  - Parallel Schwarz
  - Alternating Schwarz
- **Parallel preconditionning** *(to be used with e.g. conjuguate gradient or GMRES)*
  - Additive Schwarz
  - Multiplicative Schwarz